

Increasing the learning outcome in mathematics by use of computer programs like Mathematica

Ivar Johannesen

Assistant Professor, Oslo University College, Faculty of Engineering, ivar.johannesen@iu.hio.no

Abstract - Since early nineties I have been an enthusiastic user of Mathematica in teaching and research. During this period the program has improved tremendously in every new release and the power and speed now available leaves this program the most professional tool for computational mathematics. In addition to simple calculations, Mathematica offers advanced programming techniques and very fast numerical algorithms. It is very easy to show impressive visualisations of any concept relevant to engineering students. Unlike other numerical programming tools Mathematica can do symbolic calculations and give exact answers to complicated problems.

Index Terms – Algebra, Calculus, Computer aided teaching, Mathematica programming,

INTRODUCTION

In this paper I will tell about my experiences with this programming tool, show examples from my teaching experience and discuss the feedback from students in different study programs. After a short introductory course teaching the syntax of the program the students have used it in project work in first level calculus courses, I have also tried to motivate the student with programming examples in courses in hydrodynamics and indoor environment problems for the environmental engineering program and reaction kinetics in the chemistry program. Illustrating animations are easily obtained to model oscillations and vibrations in demonstrations in the construction engineering and electrical engineering programs. In this paper I will concentrate on mathematical concepts which are suitable for explorations in a computer lab.

Teaching mathematics is often associated with tedious calculations on whiteboards, trying to get the students grasp the essence of the topic at hand. The prerequisites are often not fulfilled when the freshmen enter college, and there are needs for lot of repetitions of basic skills. My experience is from teaching forthcoming engineers at Oslo University College. During the first two years the young students take part in calculus courses, working hard with differentiation and integration problems, then turn on to linear algebra and complex numbers, differential equations and convergence criteria of sequences and series. The curriculum will be

experienced as more or less relevant for their field of interest, and the motivation factor will play an important role in their ability to follow the syllabus.

During the last years there has been a growing interest in integrating pure mathematical topics into the special courses the students attend. Every teacher has been confronted with these questions: “Why are we learning this? What are the significance to my study program? Will I ever use this knowledge as practicing engineer?” Questions like this can be hard to give an honest answer to. There are good reasons for the students to ask for the relevance in their program, but the answer is rather vague in the student’s ears. Mathematics constitute the basis for all scientific disciplines, calculations lie at the bottom of most study work. A good understanding of mathematics enables the student to think clear and logical and make abstractions from the present situation.

One way of attacking the relevance problem is to incorporate more mathematics in the special study courses, and increase the credits for those courses. As university graduates we feel more relaxed when experienced mathematicians and physicians take hand of the lecturing of applied mathematics than leaving this to the engineering staff. Our experience has been that such efforts often have faded away, the courses remain mainly unaffected of the additional material. Students are given credits twice for the same proficiency. Solving physical problems will necessarily involve doing basic mathematics, but it must be wrong to give extra credit for this part of the course. Including mathematical subjects in a physics courseware must involve some new elements not credited before.

COMPUTERS AS AN EDUCATIONAL TOOL

Another approach is to look closely into the contents of the study courses, and to take this knowledge back to the pure mathematics courses. In close collaboration with the study lecturer we design an applied mathematics program which goes beyond the basic skills.

The necessary proficiency in handling basic computations in mathematics must never be forgotten. Pure mathematics must lie in the bottom of every study program in engineering education. The students must be drilled in skills such as factorization and simplification of mathematical expressions, and how to differentiate and integrate simple functions. Then they can use this knowledge in applications, such as

calculating areas, arc lengths and curvature of functions. Civil engineers need to calculate volumes, centroids and moments of inertia of bodies. Electrical and mechanical engineers will deal with Laplace transforms and Fourier analysis. For the chemical and environmental engineers, partial differentiation is essential in the study of thermodynamics. This list could be extended forever, but the idea here is that after basic skills are established, students should be challenged with more advanced and applied examples.

During the last decade the computers have been much faster and extremely more powerful giving new perspectives of how complex problems students can deal with. With a mobile computer connected to wireless Internet the students can do complicated calculations when and where he wants, and online help and information are always at hand.

The teaching of subjects in study programs should reflect the new situation. Since 1990, I have had the opportunity to use Mathematica, established in 1988 by Stephen Wolfram. This has evolved to be a world wide, leading mathematical software suitable for both scientific and educational work. Mathematica is able to give the results of internal computations in symbolic form. This gives us the opportunity to explore topics in depth, varying parameters to see the effect of these changes. The solutions can be visualized in a number of ways elucidating the concept under study. Other programs like Maple and MatLab will give us much the same opportunities to explore mathematics, but MatLab is essentially a numeric tool and do not possess the symbolic strength of Mathematica.

As a first example let us explore the concept of composite and inverse functions.

WORKING WITH COMPOSITES

Composite functions are rich in both mathematical content and structure yet are often presented at a superficial level. One reason for this may be attributed to the fact that the study of composite functions requires, at the outset, a more-than-adequate understanding of the functions itself. The traditional approach is to construct composite functions using contrived, trivial functions thereby emphasizing procedural and mechanical concerns over conceptual ones. Specifically, students are typically given very simple function $f(x)$ and $g(x)$ and are asked to find $f(g(z))$ for some real z . Consider the following which typifies a student's introduction to composite functions.

“Given $f(x) = x+1$ and $g(x) = x^2$,
Calculate $f(g(1))$ and $g(f(-3))$ ”

Such an approach diverts the student's attention away from the crucial idea of composite functions. Namely, that elements from a given set will be experiencing back- to - back mappings or rules, which if one wishes, may be written as a single mapping in its own right.

Specifically, students using Mathematica can easily create and work with more sophisticated functions than linear or quadratic functions. In this instance, the functions created are composite functions and the purpose of creating them includes learning how composites look in comparison to the original functions as well as identifying the properties of the originals that are retained. For example, using the functions $f(x) = x^3 - 3x^2 - 11x + 4$ and $g(x) = x^3 - 2x + 3$, we plot the functions together with the composite function in the same graph.

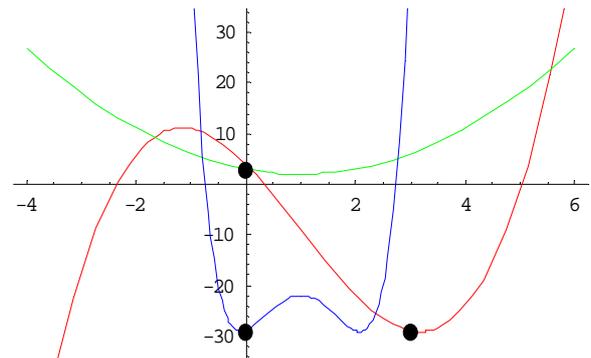


FIGURE 1
COMPOSITION OF FUNCTIONS

The teacher must explicitly demonstrate, by pointing to the graph, how the mappings $x \mapsto g(x) \mapsto y = f(g(x))$ translates the abscissa $x = 0$ to $y = -29$ through $g(0)$ and $f(3)$.

A major instructional goal of approaching the study of composite functions in such a way is to demonstrate to students that mastery of this concept will require more than the simple recall of arithmetic procedures. Specifically, mastery of this concept will require the ability to perform mechanical procedures as well as the ability to discuss underlying mathematical structure.

INVERSES OF FUNCTIONS

An important special case of composition is inversion. Within a study of functions the students are usually given an overview of how to find the inverse of a given function. However, textbooks and teachers usually ask students to only find the inverses of linear functions. This is due to the fact that the process and mechanics involved in finding those of higher degree can be intimidating and out of reach for the student. For a linear function, there exists a unique inverse function over the whole domain in consideration. If $y = f(x)$ is linear, solving the equation with respect to x will give another linear function, $x = g(y)$. Changing the independent and dependent variables in g and plotting $y = f(x)$ and $y = g(x)$ together in the same graph, will reveal to lines mirrored about the line $y = x$. The students are challenged to calculate $f(g(x))$ and $g(f(x))$ and thus verifying that $g = f^{-1}$. For functions of higher degree one must restrict the domain to an interval where the function is strictly monotonous. In such a domain there will be a unique solution to the inverse function. Over a general domain the

inversion graph will be composed of curves belonging to different inverse functions, each defined in a restricted, non-overlapping domain. Plotting such complex functions together reveals the importance of monotony of functions. With a computer tool the students can explore functions of higher degree with ease and see the resulting graphics with just a few lines of code.

Let us look at an example of a quadratic equation and let Mathematica solve the inversion problem before printing the original function together with the inverse functions. With the command `ImplicitPlot`, we can visualize the whole graph belonging to both branches of the inverse function.

Given $h(t) = -16t^2 + 10t + 5$, solving for t gives two branches

$$t = \frac{1}{16}(5 \pm \sqrt{105 - 16h})$$

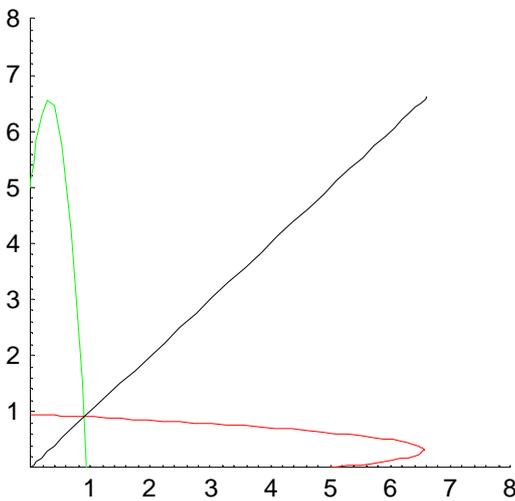


FIGURE 2
INVERSE QUADRATIC FUNCTIONS

The students should discuss what part of the red graph belongs to which branch. The one-to-one relation in the very concept of function contrary to one-to-many is crucial in understanding mathematical functions. A further exercise is to calculate the identity relation $h(h^{-1}(x))$, which is not trivial. The following code will check this.

```
h[t_] := -16 t^2 + 10 t + 5
Solve[h[t] == y, t]
invh1[t_] := (5 - Sqrt[105 - 16 y]) / 16
h[invh1[t]] // Simplify
invh2[t_] := (5 + Sqrt[105 - 16 y]) / 16
h[invh2[t]] // Simplify
```

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Continuing the investigation, some students will suggest trying out a third order function, with three branches. With $h(t) = -2t^3 + 5t^2 + 1$, the real-valued branch will be given by the expression

```
invh[t_] := (1/6) * (54 y + 6 Sqrt[3 - 27 y^2 - 179 y + 152] - 179 + 5 - Sqrt[3 - 27 y^2 - 179 y + 152] - 179)
```

Plotting $h(t)$ and all branches of $h^{-1}(t)$ in the same graph:

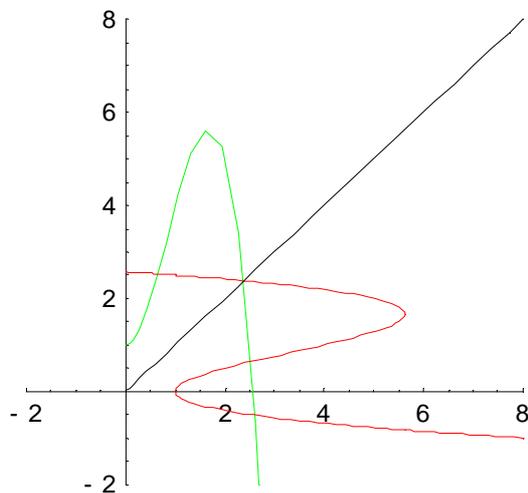


FIGURE 3
INVERSE CUBIC FUNCTIONS

The three branches are clearly detectable from the graphics. This will not be possible to show without a computer program to do the calculations and visualize the graphs. It is also easy to check the correctness of our inverse function $y = \text{invh}(t)$:

```
h[invh[t]] // Simplify
t
```

ARC LENGTH OF ELEMENTARY FUNCTIONS

In a calculus course much time is spent on calculations of areas under graphs and volumes of surfaces of revolution. Typical textbooks treat and assign arc length problems that seem artificial, while avoiding more interesting cases such as the arc length of an arch under the sine or the arc length of an ellipse. This is done because the latter don't lead to integrands with elementary antiderivatives. Only when the integrand simplifies to an exact square, does the integral evaluate in a simple manner. The fundamental theorem of calculus will

nevertheless guarantee that any appropriate sequence of Riemann sums leads to an approximation of any desired accuracy (at least for functions with continuous integrands).

With a computer program at hand one will always be able to compute a numerical approximation of the arc length. But Mathematica offers extraordinary symbolic power which gives us analytic expressions for the arc length ready for plotting. Consider the parabola $y = x^2$ with arc length element equal to $ds = \sqrt{1 + 4x^2}$. Integrating this from 0 to x gives us an answer consisting of an inverse hyperbolic function. For our purpose, this is quite satisfactory, but the students ought to have a quick survey of logarithms before continuing.

$$s = \int_0^x \sqrt{1 + 4x^2} + \frac{1}{4} \text{ArcSinh } 2x$$

Using the fundamental theorem, we can calculate the arc length from $x=0$ to $x=1$:

$$s = \frac{5}{2} + \frac{\text{ArcSinh } 2}{4}$$

A numerical approximation of the length of this arc is 1.4789. Moreover, we can get an overview of how the arc length increases with increasing x :

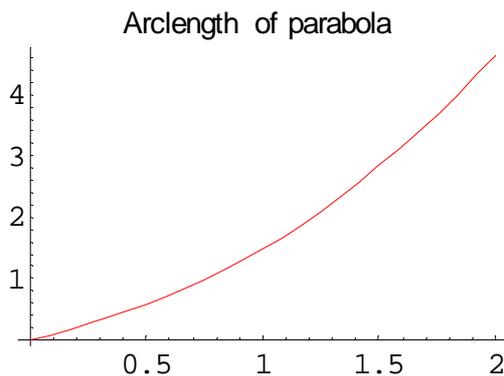


FIGURE 4

Working in a Mathematica lab, students are encouraged to investigate their own functions. Often the symbolic answer will include advanced functions not known to students, but still the main goal is achieved: reaching an analytic expression for the arc length which can be plotted and compared to other functions. With $y = x^3$ we arrive at hypergeometric functions, a general type of functions of which many elementary functions are special cases.

$$\int_0^x \sqrt{1 + 9t^4} = {}_2F_1\left(-\frac{1}{2}, \frac{1}{4}; \frac{5}{4}; -9x^4\right)$$

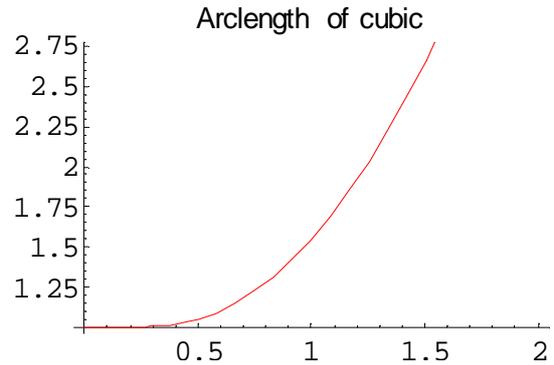


FIGURE 5

Comparison can be made between the two graphs. Then we discover something interesting: In the beginning the arc length of the parabola exceeds the arc length of the cubic functions, but after approximately $x = 0.9$ the cubic wins.

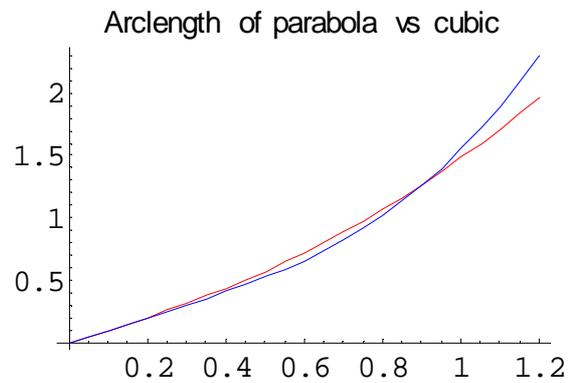


FIGURE 6

If we want to pursue this observation further, the students can do a numerical calculation for finding the highest x -value where the two arcs have equal length.

$$\text{FindRoot}\left[\frac{x}{2} \sqrt{4x^2 + 1} + \frac{1}{4} \text{ArcSinh } 2x - \frac{x^5}{5} + \frac{5}{4} x^4, \{x \rightarrow 0.909033\}\right]$$

Thus we find the answer $x = 0.909$ to our problem.

Next we turn to the problem of finding the length of the perimeter of an ellipse. The standard form for an ellipse in rectangular coordinates is $(x/a)^2 + (y/b)^2 = 1$. Here a and b represent the half-axes of the conic section. Introducing the eccentricity $e = c/a$, where c is half the distance between foci, we solve the equation with respect to y in the following way:

$$y = \sqrt{(1 - e^2)(a^2 - x^2)}$$

Integrating the arc length from $x = 0$ to $x = a$ will give us one quarter of the perimeter due to symmetry. The length of the full the perimeter therefore becomes

$$4 \int_0^a \sqrt{1 + \frac{(1 - e^2)x^2}{a^2 - x^2}}$$

The eccentricity e varies from 0 to 1, the lower value represents a circle since the two foci coincide when $c = 0$. When e approaches 1, the ellipse degenerates to a line, and the perimeter will be twice the length of the line. The parameter a just scales the whole result proportionately, so we can fix the value to $a = 1$. We can illustrate how the circumference decrease from 2π to 4 as the ellipse collapses to a straight line.

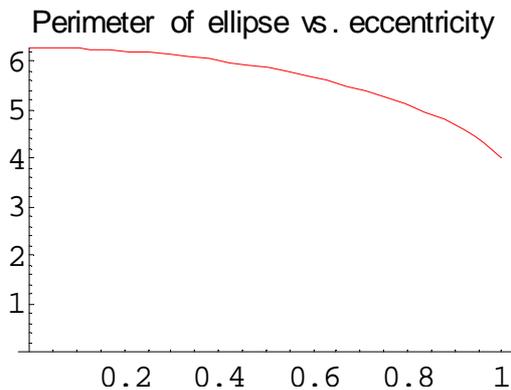


FIGURE 7

Next I turn to an example from teaching of series, a topic that really needs stunning surprises to awaken the students.

VISUALIZING ALGEBRAIC CONCEPTS

As educators, we hope that the use of software such as Mathematica will allow students to explore, experiment, and perhaps discover ideas and results that are new to them. Here I will present my favourite application from the theory of conditionally convergent series, which may fascinate the most bored student. First a story from the harmonic series – divergence may also be of interest.

Teaching convergence criteria for series to engineering students is a tough experience. Somehow the concept of sequences and series is out of reach for many of the engineering students. As logical as it may appear to the tutor, comparison tests feel very useless when the freshmen see no clues to what series to compare with. There are a lot of simple tests to try, but often only one of them is adequate. Integral test is nice to do when the convergence of the corresponding improper integral is easily decided, but when the calculation

of the integral itself becomes a problem, the task seems tremendous. The harmonic series is the violation, since the divergence of the logarithmic integral is obvious. It is a very important result, since the harmonic series compares to many divergent series. So the students get it hammered in: the harmonic series diverges, and at a very slow pace. But there is a lot more to say, and with Mathematica accessible, there is no excuse for not doing so. First let the students define a new sequence, consisting of the partial sums of the harmonic series, and make a comparison between the general term in this sequence (the harmonic number H_n) and the natural logarithm. As n increases, the difference between terms seems to settle down to a constant value. Therefore, the term by term subtraction of two divergent sequences may possibly give a new convergent sequence.

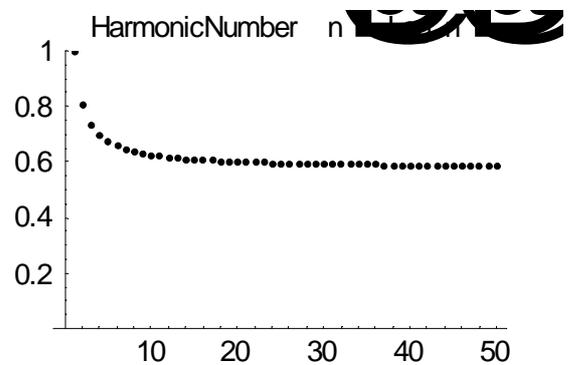


FIGURE 8

From here, we can let the program introduce the students to a new irrational number:

$$\lim_{n \rightarrow \infty} H_n - \ln n = \gamma$$

Here $\gamma = 0.577216$ is The Euler - Mascheroni constant. In fact the constant has never been proven to be irrational, although a lot of efforts have been made to solve this persistent problem. More than 100 million decimal digits are known from time-consuming computer search. The interested reader will find more information about this constant on the website [3]. So much for series with positive terms, where integral test and comparison tests apply. When you get to series with negative terms mixed in, the same tests do not apply any more, and subtle concepts as absolute and conditional convergence enter the scene. Usually, when the series include negative terms, the subjects of teaching restrict to alternating series, where Leibniz test for convergence is the students only choice of test for conditionally convergent series. Studying monotony of the absolute value of terms will guide the students to the correct answer, but still there is a mysterious shimmer over those series. What does the word conditionally mean in this context?

The engaged and knowledgeable tutor will now tell a story hard to believe for the students. The conditionally convergent series can converge to different values according to the way you order the terms! In fact, any value is possible! This sounds like magic for the audience. Some textbooks take a notice of this fact, and give an example with the alternating harmonic series. Written in the usual form $1 - 1/2 + 1/3 - 1/4 + 1/5 - \dots$, that is positive and negative terms positioned 1:1:1:1:..., the sum is $\ln 2$, where \ln means natural logarithm. This result is easily derived from the Maclaurin series of $\ln(1+x)$, convergent for $-1 < x \leq 1$. Rearranging the terms like $(1 - 1/2) - 1/4 + (1/3 - 1/6) - 1/8 + (1/5 - 1/10) - \dots = 1/2 - 1/4 + 1/6 - 1/8 + 1/10 - \dots$, we immediately see that the sum is halved to $1/2 \ln 2$. The pattern between positive and negative terms is now 1:2:1:2:.... Rearranging again, this time with the pattern 1:4:1:4:1:..., removes the other half and the sum converges to exactly 0. But as told, this is not the end of the story. Given any reasonable number, say between -10 and 10, you can always rearrange the terms so the new series converge to your number. Ed Packel and Stan Wagon [1] have written a small program which does exactly this. The program code is available at my home site [2]. In the classroom, I ask for a number from the audience, runs the application for a second or two, and –voila– the right pattern for obtaining the requested number enters the projector screen. For instance, if you want the series to converge to π , you must arrange the positive and negative terms in the sequence 76:1:129:1:132:1:134:1133:1:134:1:134:1:133:1...

EVALUATION BY STUDENTS

How do the students respond to working with computer projects in their learning situation? To answer this question, it is necessary to look at the whole curriculum the students have to deal with during three intense, hardworking years. The first two years contain three major courses in mathematics and physics, in addition to study courses in the bachelor program. Their main interest lies in the study courses, and doing mathematics is a necessary and often painful extra endeavour to reach their final goal.

Using a large professional program like Mathematica might seem like overkill, and many fellow colleagues share this opinion. They have one thing in common, they have never tried out the program themselves. There is no doubt a conceptual barrier which students have to overcome to take advantage of the computational power of the program. Therefore we focus very much on the introductory part, emphasizing the importance of strictly accepting the syntax and working out simple calculations easy to check. We are two instructors co-working in the lab with about 20- 25 students. The program comes with tutorials, on line manual and help browser, and Wolfram Research supports web sites probably being the most comprehensive source of mathematics and natural science you will ever find. So there really is no excuse not attacking the difficulties appearing during work.

The earlier the students get to know Mathematica, the more positive they become to use this tool as a real help in their line of study. Starting first in their last year, the benefits of the computer program are small. Indeed, they will possibly not use this program in future work as engineers, because it is mainly a professional tool for scientists and researchers at universities and research laboratories. But their proficiency in mathematical software will make it easier for them to use any computational tool in their practice, and today's engineer must be prepared for a more computerized working environment than their predecessors.

The attitudes of the students to computer lab also vary with study programs. In some classes there is enthusiasm and joy, and the students encourage each other to solve challenging problems. They spend a lot of time improving the layout of their reports, and make beautiful graphics with colors, line styling and explanatory text. In other classes the usefulness of computer projects are not fully emphasized, mathematics is difficult on its own and dealing with syntactic rules are another burden to cope with. Projects are time consuming and the subjects in the course syllabus need full concentration at basic level. Many teachers hesitate to get involved in new educational tools requiring their spare time.

CONCLUSIONS

I have tried to illustrate with my examples in this paper, my need for a computational tool going beyond just an advanced calculator. I appreciate the program's symbolic power and visualization capabilities, and the numerical speed due to very fast algorithms. The students gain insight in sophisticated problems they would never manage on their own, and it gives inspiration to further explorations in the topics.

In a forthcoming paper I want to give other examples of my teaching experience with Mathematica, also in areas of physics, chemistry and indoor climate in the environmental engineering program. The latter was also touched upon in my paper to the ICEE 2006 conference in San Juan, Puerto Rico [4]

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