Euler’s tables, the origins of structural analysis, and science’s lingua franca in the Enlightenment

Anísio Andrade[1], Filipa Medeiros [2], Nair Soares [3], Paulo Providência [4]

***Abstract* - The aim of this paper is threefold: (i) to elucidate the early history of the displacement (or stiffness) method of structural analysis, (ii) to draw attention to the large body of important scientific texts that remain concealed behind “forgotten” languages such as Latin, and (iii) to illustrate how the creation and early growth of a concept may be relevant to the teaching of engineering disciplines. We argue that Euler’s relatively unknown paper on the “problem of supports”, written in Latin, establishes him as the creator of the displacement method. Moreover, we show that the approach used to introduce the displacement method in typical textbooks on strength of materials is remarkably close to Euler’s procedure.**

Key Words- Displacement method, Euler, History of science in engineering education, Latin, Problem of supports

Lisez Euler, lisez Euler, c’est notre maître à tous.
PIERRE-SIMON LAPLACE

INTRODUCTION

Every teacher of science and technology is faced with the difficult task of deciding on the best way of presenting to his students, for the first time, fundamental concepts and methods that are by no means obvious. The inception and early unfolding of these concepts and methods may often be of considerable pedagogic value, since they then appear reduced to their very core, or crystallized in special cases, yet “containing all the germs of generality” (to quote D. Hilbert). We illustrate this viewpoint by considering the approach used to introduce the displacement (or stiffness) method of structural analysis in typical textbooks on strength of materials, and by showing that it is remarkably close to the one used by Euler in his memoir *De pressione ponderis in planum cui incumbit* (On the pressure exerted by a weight on the plane on which it rests), presented to the St. Petersburg Academy on March 22, 1773, and published in the following year [1]. To do so, we first present an analysis of this ingenious and relatively unknown work, where we try to unravel the doubts, perhaps even misinterpretations, that it has caused among some of the most eminent historians of structural mechanics. Based on this analysis, we claim that Euler should be credited, in plain justice, with the creation of the displacement method. We also wish to draw attention to the immense body of scientific literature that remains concealed behind “forgotten” languages such as Latin – to make it again a living branch of our cultural heritage is a challenge worthy of the exertions of men of science, historians and philologists.

This paper is humbly dedicated to the memory of Leonhard Euler, on the occasion of his tercentenary.

THE BIRTH OF THE DISPLACEMENT METHOD – EULER’S ANALYSIS OF THE “PROBLEM OF SUPPORTS”

Historical uncertainties

The displacement method is one of the fundamental methods of structural analysis and is covered in every undergraduate course in civil, mechanical or aerospace engineering. When, how and by whom was this method created? In a recent review paper [2], Samuelsson and Zienkiewicz, two leading figures in the field of computational mechanics, trace the origins of the displacement method back to Clebsch’s treatise on the theory of elasticity [3, 4], with a fleeting reference to Navier’s lecture notes of 1826 [5] (in fact, as pointed out by Pearson [6, p. 146], this matter had already been considered by Navier in his lectures for 1824 and in a note contributed in 1825 to the *Société Philomatique de Paris* [7]). However, according to Timoshenko [8, p. 36], the first treatment of a statically indeterminate problem is to be found in [9], a paper taken from Euler’s notebooks by Jacob Bernoulli II (who was presumably unaware of the existence of [1]) and published in German by his brother Johann Bernoulli III in 1795 – an English translation of this document is available in [10]. But Timoshenko does not link Euler’s name to the displacement method, and neither does Benvenuto, who first uses the expression “deformation method” when discussing Clebsch’s work [11, p. 492]. Moreover, in his analysis of the memoir [1], Benvenuto casts some doubts on the soundness of Euler’s reasoning, suggesting that even if he had found an answer to the “problem of supports” (what reactions occur when a body is supported at more than three non-collinear points or at more than two collinear points?), he wouldn’t be able to explain why the answer was correct [11, pp. 442-444]. We address these doubts in the remainder of this section, and we argue that [1] truly establishes Euler as the creator of the displacement method and the year 1773 as its birth date. But before we proceed, one further question remains: why is this important work so little known today? The answer lies in a combination of factors: firstly, it is written in Latin, an almost forgotten language in scientific and engineering circles; secondly, the displacement method really came into its own as a tool for the analysis of skeletal structures, while the “problem of supports” was not particularly

[1] Anísio Andrade, Department of Civil Engineering, University of Coimbra, anisio@dec.uc.pt
[2] Filipa Medeiros, Institute of Classical Studies, University of Coimbra, filipa-medeiros@clix.pt
[3] Nair Soares, Institute of Classical Studies, University of Coimbra, nairncs@ci.uc.pt
4 Paulo Providência, Department of Civil Engineering, University of Coimbra, provid@dec.uc.pt

relevant for engineers; thirdly, the immense bulk of Euler's publications and his many other achievements in all fields of mathematics and mechanics might have eclipsed this one.

A giant leap forward – The birth of the displacement method

To discuss the birth of the displacement method, first we must decide what our present-day understanding of this method is. Therefore, we begin this sub-section by listing its distinctive features (a salutary pedagogical exercise in itself):

(i) The displacement method is applicable to the analysis of kinematically indeterminate structural systems. The degree of kinematical indeterminacy of the system is the number of independent parameters needed to define its deformed configuration. These parameters are called the generalized displacements.

(ii) There is an equilibrium equation associated with each generalized displacement.

(iii) The governing (or canonical) equations of the method, with the generalized displacements as unknowns, combine equilibrium, kinematics and constitutive equations. They are obtained by inserting the constitutive equations (written in stiffness format) and the kinematical conditions into the equilibrium equations.

We add a fourth, non-fundamental, feature to this list, since it is intrinsic to the actual use of the method:

(iv) Once the generalized displacements are known, the kinematical and constitutive equations can be used to find the internal forces. This is usually referred to as the post-processing phase.

How does Euler's memoir fit in this conceptual framework? He tackles a very specific problem: that of finding the forces or pressures (the two words are used synonymously by him) exerted by a weighting body on a supporting medium with a horizontal surface. If the body rests on three non-collinear point supports, the problem is statically determinate, and equilibrium alone is enough to solve it. When the body rests on more than three non-collinear supports, or on a continuous base, the problem becomes statically indeterminate – equilibrium considerations alone are no longer enough to render a unique solution. (From an abstract viewpoint, this amounts to the decomposition of a force along more than three parallel directions not contained in the same plane.) Any attempt to arrive at a unique solution using purely statical means is doomed to failure – but this was not known in Euler's days, and many tried to follow that path, as Benvenuto so vividly describes in [11, pp. 447-460].

Euler, on the other hand, shows that it is possible to reach a definite solution if we (i) take into account the deformations of the system (kinematics), and (ii) accept a causal relationship between forces and deformations (constitutive relation). He admits that the body is (perfectly) rigid, but the supporting medium is flexible, and this according to a linearly elastic law. Kinematical considerations tell him that the base of the body still defines a single plane after deformation, and this plane is characterized by three independent parameters – the generalized displacements of the modern terminology. If the foundation is homogeneous, the combination of this kinematical result with the constitutive assumption of linear elasticity leads to the conclusion that the forces exerted by the body in the foundation also define a single plane. Euler calls this his *General Principle*. Using this *principle*, the three equilibrium equations are now written in terms of the three independent parameters characterizing the plane of the forces, which Euler takes as unknowns. This yields a unique solution to the problem. We should also mention that Euler is careful and meticulous enough to assume small vertical displacements, so that he can adopt linear kinematical and equilibrium equations.

From this brief description, we plainly see that Euler's procedure meets the basic requirements to be considered a (specialized version of the) displacement method (tailor-made for the specific problem under consideration). Indeed, (i) Euler identifies three generalized displacements and the corresponding equilibrium equations; (ii) he merges kinematical and constitutive relations into a single principle; (iii) his governing equations are obtained by inserting this principle into the equilibrium equations. This combined use of statics, kinematics, and constitutive assumptions is truly remarkable. In a sense, it is a change of paradigm in the sense given by Hall [12]. One point deserves further comment. Euler's unknowns are not the generalized displacements, but the parameters defining the plane of the forces. From a mathematical viewpoint, this is merely a change of variables – we easily recognize the parameters defining the plane of the forces as the generalized displacements scaled by the stiffness of the foundation. We must also bear in mind that Euler's goal is to find the forces exerted on the support. Therefore, he proceeds directly to what we call today the post-processing phase, bypassing the determination of the displacements. Moreover, the displacements depend on the stiffness of the foundation, whereas the forces do not – setting up the unknowns as he did, he avoided the need to assign a definite value to this stiffness.

But is Euler really sure of the solution he is proposing for the "problem of supports"? Is he able to describe the path that leads him to the solution and to explain why it is correct? According to Benvenuto, he is not [11, pp. 442-444], since he writes that his solution remains valid in the case of a rigid foundation – therefore, he does not regard the deformable foundation as a fundamental feature of both the problem and the solution method he devised. We believe this is a misinterpretation of the Eulerian text. To settle the question, we need to look more closely at §§ 4 - 5 of Euler's memoir, where he lays down the foundational ideas of his method. These paragraphs are at the heart of Benvenuto's criticism.

Reading Euler's Latin – Did he really know what he was talking about?

After briefly describing an elegant geometrical solution to the statically determinate tripod problem (a weight resting on three non-collinear supports), Euler attacks the *"much more difficult"* case of four supports, whose solution "*seems entirely indeterminate and slippery."* At the beginning of § 4, he introduces the notion of a deformable foundation. Indeed, a rigid foundation precludes a unique solution to this problem, although we must acknowledge that Euler nowhere makes such a statement explicitly:

"… we shall conceive that the plane or soil on which the weight rests is not so rigid that it cannot be impressed upon, but instead that it has been covered, so to speak, with a cloth, into which the feet can slightly penetrate."

The cloth is introduced as an analogy, an image with increased visual impact to convey the fundamental notion of a deformable soil – while the impressions on the soil may be imperceptible, and thus hard to imagine, the deformation of the cloth is easily visualized. Notice that Euler carefully chooses his words: he writes *"quasi panno obductum"* (*"covered, <u>so to speak</u>, with a cloth"*).

Euler proceeds by postulating a linearly elastic law for the foundation:

"… it may be safely assumed that the [depth of the] *impression under each foot will be proportional to the force exerted on the soil and, once this principle is accepted, all the work can be easily explained."*

Benvenuto argues that Euler seems to regard this law as "self-evident," and not as a "specific hypothesis about the behavior of material under stress" [11, pp. 443-444]. But we must also consider that linear elasticity is the simplest constitutive assumption that he may adopt in order to solve the problem, and Euler's eager desire to arrive at an easy explanation of his method is manifest throughout these first paragraphs of the memoir.

Euler closes § 4 with the following remark:

"However, so that no one is troubled by this cloth yielding to pressure, even though we have assigned softness to it, we may nevertheless diminish this softness as much as we like, so that the nature of that soil on which the weight actually rests is finally reached."

Benvenuto's critical appraisal of the memoir rests mainly on his interpretation of these words. According to him, when Euler writes *"indolem soli illius"* (*"the nature of that soil"*), he is referring to a rigid soil, and therefore the cloth's softness may be diminished until it vanishes. Benvenuto infers from this passage that "the assumption of a soft plane is [to Euler] only a conceptual device, useful for understanding what happens in a rigid plane, and the transition from one case to the other is ruled by a common argument in mathematics," a passage to the limit [11, pp. 444]. We cannot find in Euler's text objective grounds for such a conclusion. Instead, we believe that Euler is referring to the soil *"not so rigid that it cannot be impressed upon"* of the preceding sentence, whose deformability, however, is much smaller than that of the imaginary cloth. Thus, Euler nowhere writes that his solution remains valid in the limiting case of zero flexibility of the soil. On the contrary, a deformable foundation is a fundamental feature of both the problem and the solution method he devised.

In § 5, Euler writes:

"Let us then consider four feet, the ends A, B, C, D *of which stand on the plane, and, upon bearing, penetrate that cloth by small distances* Aα, Bβ, Cγ, Dδ, *which must be assumed as infinitely small. This being laid down, firstly the points* α, β, γ, δ, *like the ends of the feet, will still be located on a single plane; secondly, these small distances* Aα, Bβ, Cγ, Dδ *are taken as being proportional to the pressures exerted on the soil by each foot. Therefore, if at the points* A, B, C, D *we erect on the plane vertical segments* Aα, Bβ, Cγ, Dδ *that are proportional to the pressures at those points, it is necessary that the points* α, β, γ, δ, *lie on a single plane. And this is the principle upon which we may safely build our entire investigation, all the more so because neither the idea of that cloth nor the impressions made upon it are taken into account any longer; indeed, these ideas were invoked solely for the sake of aiding our reasoning."*

Euler begins this paragraph by assuming that the vertical displacements of the feet are very small, an hypothesis that will allow him to linearize the equilibrium and kinematical equations, and that is physically justified by the minute deformability of the soil. He then establishes *"the principle upon which we may safely build our entire investigation."* In doing so, Euler combines kinematics (*"points* α, β, γ, δ, *like the ends of the feet, will still be located on a single plane"*) with constitutive assumptions (*"these small distances* Aα, Bβ, Cγ, Dδ *are taken as being proportional to the pressures exerted on the soil by each foot"*), and since he is tacitly assuming that the constant of proportionality is everywhere the same, he concludes that the reactions under each foot define a plane. (In §6, Euler generalizes this result to a *"plane base of arbitrary figure"* and calls it his *"General Principle."*) Euler's remark at the end of §5 is again at the heart of Benvenuto's criticism. Indeed, according to this Italian scholar, Euler is stressing that his principle, "though obtained from the example of a yielding surface, is independent of it, because the hypothetical displacements *«have been introduced solely for the sake of aiding our imagination»*" [11, p. 444]. In other words, Benvenuto believes that Euler is once again denying the inescapable need to take into account the deformability of the supporting medium. We consider this to be a misinterpretation of Euler's words. In our opinion, what he is in fact saying is that, after having established his principle, he does not need to keep bringing up (*"venire in censum"* – literally, *"to record on a list"*) the image of the cloth, which was just a simile, an analogy, used to illustrate his thought and to help his readers grasp something that was, at the time, a conceptual innovation (*"in subsidium nostrae imaginationis"*). Moreover, the *General Principle* into which he has merged kinematics and the constitutive response of the soil can be subsequently used as a black box and inserted directly into the equilibrium equations – as we have seen earlier, this is a distinctive feature of the displacement method. If we now recall that the problem to be solved was that of finding the forces exerted on the support – Euler is not interested in the magnitude of the displacements *per se*, as long as they are small –, it is readily recognized that the precise (but finite) value of the soil's uniform stiffness is immaterial.

The irony in all this is striking: the cloth that Euler meant solely as a visual aid turned out to be, two centuries later, greatly responsible for the misinterpretation of his memoir and the failure to grasp its full significance. The irony is all the more striking because Jacob Bernoulli's transcript of Euler's notebooks [9] does not contain a single reference to this imaginary cloth. This argument is by no means decisive, and we build our entire case without alluding to it, since it is not possible to assess the nature and extent of Jacob Bernoulli's editorial work without direct access to Euler's notes.

One final remark: even if we accept Benvenuto's interpretation (which we do not), we must grant Euler the priority in creating the displacement method. According to Truesdell, "the first principle of historical research in mechanics [is] that the meaning is to be inferred from the use, since successful application has always preceded statement of the principle being applied" [13, p. 244]. Euler not only devised a solution method to the "problem of supports" that exhibits the basic features of the displacement method (even if we concede that he could not satisfactorily explain how and why it worked), he also successfully applied it to several examples.

The displacement method catches the train

Two subsequent milestones in the early development of the displacement method deserve to be briefly mentioned here.

After the publication of Euler's memoir, the next significant contribution to the development of the displacement method is to be found, half a century later, in Navier's celebrated lecture notes [5]. Navier applies the method to the analysis of the plane trusses shown in Figure 1, and this represents a distinct progress – the focus moves from the support reactions on a solid body to the internal forces in the bars of a framework, an important and pressing engineering problem at a time of rapid expansion of the railway networks in Europe and North America. Another important development is the consideration of different "elasticity forces" (*i.e.* different flexibilities) for each bar. Equally important is the fact that Navier openly states the combined use of statics, kinematics and elasticity.

The complete systematization of the displacement method for truss analysis, allowing for any number of nodes and bars, was accomplished by Clebsch in section 90 of his treatise of 1862 [3, 4]. He introduced stiffness coefficients and their determinants, foreshadowing the matrix formulation of the method that would take place in the 20th century [14]. Clebsch also considered frameworks with bars subjected to bending (section 91 of his treatise), but he did not achieve in this case the same degree of closure and elegance.

LECTURING THE DISPLACEMENT METHOD TODAY

In civil, mechanical and aerospace engineering courses, students are usually first exposed to the displacement method in the discipline of strength of materials. The method is further developed in subsequent disciplines on structural analysis, usually with the adoption of a matrix formalism, and its application is extended to non-linear and time-dependent problems. Very often, the finite element method is presented as a generalization of the displacement method.

In most textbooks on strength of materials – for definiteness, we choose Dias da Silva's *Mechanics and Strength of Materials* [15], but nearly any other would serve as well, except that some are less careful to state exactly what is done and what the assumptions are – the presentation of the displacement method follows a problem-based approach. The problem with two degrees of kinematical indeterminacy shown in Figure 2, slightly adapted from [15, p. 168], is typical of such an approach. It can be rightfully regarded as a two-dimensional analog of Euler's *Problem 2*, in which he considers a weight resting on a plane by means of four point supports, arranged according to the vertices of a parallelogram – the role of the deformable foundation is now played by the props. (A horizontal support has been added with the sole purpose of properly restraining the structural model against rigid body displacement.)

Let us assume that the vertical displacements are small and the props do not buckle, so that the hypothesis of geometric linearity holds. For convenience, we define a horizontal axis along the beam, with origin at an arbitrary point *O*.

The equilibrium equations for this simple model are

$$\sum_{j=1}^{3} f_j = G \, , \; \sum_{j=1}^{3} x_j \; f_j = X \, G \qquad (1)$$

where f_j is the compressive force in prop j, G is the applied load, and x_j and X denote the abscissas corresponding to the prop j and to the load G. Incidentally, students are expected to recognize that the moment equilibrium condition is

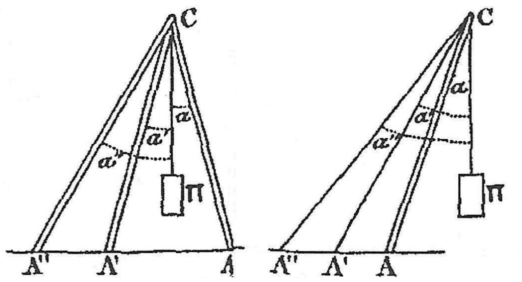

FIGURE 1
PLANE TRUSSES ANALYZED BY NAVIER [5].

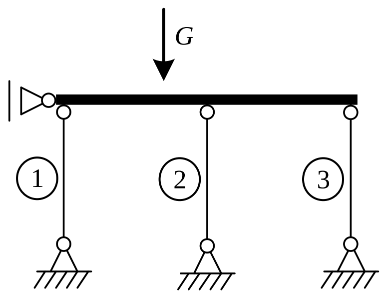

FIGURE 2
RIGID BEAM SUPPORTED BY THREE VERTICAL AND LINEARLY ELASTIC PROPS.

configuration-dependent; it is written for the undeformed configuration as a consequence of geometric linearity, a frequently overlooked detail at undergraduate level.

The force f_j in prop j is related to its axial shortening Δ_j through the linear elastic constitutive relation

$$f_j = k_j \, \Delta_j \, , \qquad (2)$$

where the proportionality constant k_j is called the stiffness of the prop.

Kinematics provides the final set of equations:

$$\Delta_j = d_j = D_1 + D_2 \, x_j \, , \qquad (3)$$

where d_j is the downward vertical displacement of the top extremity of the prop and D_1, D_2 are the two generalized displacements characterizing the final configuration of the rigid beam (the downward vertical displacement of the origin O and the clockwise rotation of the beam).

Equations (2) and (3) – constitutive relation and kinematics – may be combined to give

$$f_j = k_j \left(D_1 + D_2 \, x_j \right) = D_1 \, k_j + D_2 \, k_j \, x_j \, . \qquad (4)$$

Now, if we assume, as Euler tacitly did, that the props have equal stiffnesses ($k_1 = k_2 = k_3 = k$), then (4) reduces to

$$f_j = k \left(D_1 + D_2 \, x_j \right) = D_1 \, k + D_2 \, k \, x_j = \alpha + \beta \, x_j \, . \qquad (5)$$

Therefore, the forces in the props define a straight line, and this is precisely (a specialized version of) Euler's *General Principle*! Inserting this result into the equilibrium equations (1), we get

$$\sum_{j=1}^{3} \left(\alpha + \beta \, x_j \right) = G \, , \; \sum_{j=1}^{3} x_j \left(\alpha + \beta \, x_j \right) = X \, G \, , \qquad (6)$$

wherefrom the unknowns α, β (generalized displacements D_1, D_2 scaled by the stiffness coefficient k) are readily found. Finally, substituting the values of α and β back into (5) yields the forces f_j in the props, which do not depend on k.

We may generalize the above problem in a number of ways. The immediate one is to consider different stiffnesses for the props. Euler's *General Principle* no longer holds, but the basic features of his solution procedure remain entirely valid. In this case, the forces f_j depend on the ratios of the stiffness coefficients k_j, but not on the absolute magnitudes of these quantities.

Another generalization, in the spirit of Euler's *General Problem*, is to replace the props by a continuous linearly elastic foundation, with stiffness k per unit length. This is known today as Winkler's model and is shown schematically in Figure 3. We see that the problem changes its form – the rigid beam no longer stands on a finite number of point supports, but rests instead on a continuous base that we may view as consisting of infinitely many linearly elastic vertical springs. Nevertheless, the degree of kinematical indeterminacy

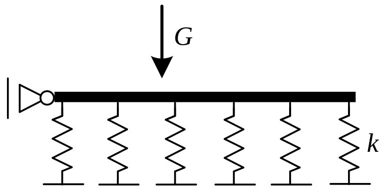

FIGURE 3
RIGID BEAM ON A LINEARLY ELASTIC FOUNDATION.

still equals two and the solution procedure outlined above requires only minor adaptations – basically, the summations have to be replaced by integrals, as Euler acutely observes in § 12 of his memoir.

Finally, returning to the problem shown in Figure 2, we could consider, in addition to the applied force G, indirect actions such as temperature changes, settlement of supports or prestress in one of the props – these indirect actions do not appear in Euler's memoir, but they can be accommodated in a straightforward and natural way.

These examples illustrate a friendly and non-abstract presentation of the fundamental concepts associated with the displacement method, which turns out to be strongly related to the historical origins of the method. On this firm foundation, we can build more effectively a general procedure for the linear analysis of kinematically indeterminate skeletal structures, and not just rigid bodies connected by elastic links. We believe that the pedagogic message is clear: History of Science can be used to enhance the effectiveness of engineering education.

CONCLUSIONS

The inception and early unfolding of concepts and methods may often be of considerable pedagogic value, since they then appear reduced to their very core, or crystallized in special circumstances – but not too special, in which case we might deflect the students from the general theory rather than lead them towards it. We illustrated this point with a specific example taken from the undergraduate civil engineering curriculum – we showed that the approach typically used to introduce the displacement method of structural analysis closely follows Euler's line of reasoning in the memoir *De pressione ponderis in planum cui incumbit*. The paper includes an in-depth analysis of but a small part of this text – the one dealing with the fundamental ideas underpinning the displacement method –, accompanied by a translation of the passages that are central to our interpretation. Based on this analysis, we claim that Euler should be credited, in plain justice, with the creation of the displacement method.

A final remark: Euler's memoir contains other seminal concepts, not touched upon in the present paper – for instance, the problem of unilateral supports, of such great importance in soil mechanics and foundation engineering, and the definition of the core (or kern) of a cross-section. Euler's approach to these matters might again prove valuable in the classroom.

REFERENCES

[1] L. Euler, "De pressione ponderis in planum cui incumbit," *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae*, Vol. 18, 1774, pp. 289-329. Reprinted in *Leonhardi Euleri Opera Omnia*, 2nd Series, Vol. 9, pp. 1-34.

[2] A. Samuelsson and O. C. Zienkiewicz, "History of the stiffness method," *International Journal for Numerical Methods in Engineering*, Vol. 67, No. 2, 2006, pp. 149-157.

[3] A. Clebsch, *Theorie der Elasticität fester Körper*. Leipzig: Druck und Verlag B.G. Teubner, 1862.

[4] A. Clebsch, *Théorie de l' Élasticité des Corps Solides*, French translation by B. de Saint-Venant and A.A. Flamant, with additional notes by B. de Saint-Venant. Paris: Dunod, 1883.

[5] C. L. Navier, *Résumé des Leçons Données à l'École des Ponts et Chaussées*, Première Partie, 1826.

[6] I. Todhunter and K. Pearson, *A History of the Theory of Elasticity and of the Strength of Materials*, Vol. 1. Cambridge: Cambridge University Press, 1886.

[7] C. L. Navier, "Sur des questions de statique dans lesquelles on considère un corps supporté par un nombre de points d'appui surpassant trios," *Bulletin des Sciences par la Société Philomatique de Paris*, Vol. 11, 1825, pp.35-37.

[8] S. P. Timoshenko, *History of Strength of Materials, with a Brief Account of the History of Theory of Elasticity and Theory of Structures*. New York: McGraw-Hill, 1953.

[9] L. Euler, "Von dem Drucke eines mit einem Gewichte beschwerten Tisches auf einer Fläche. Aus den Papieren des Sel. Leonhard Euler gezogen, von Jacob Bernoulli," *Archiv der reinen und angewandten Mathematik*, Vol. 1, No. 1, 1795, pp. 74-80. Reprinted in *Leonhardi Euleri Opera Omnia*, 2nd Series, Vol. 9, pp. 407-412.

[10] V. A. Vagliente and H. Krawinkler, "Euler's paper on statically indeterminate analysis," *Journal of Engineering Mechanics –ASCE*, Vol. 113, No. 2, 1987, pp. 186-195.

[11] E. Benvenuto, *An Introduction to the History of Structural Mechanics*, Part II: Vaulted Structures and Elastic Systems. New York: Springer-Verlag, 1991.

[12] A. R. Hall, *Revolution in Science 1500–1700*. London: Longman, 1983 (Portuguese edition by Edições 70, 1988).

[13] C. A. Truesdell, "Whence the law of moment of momentum?" in *Essays in the History of Mechanics*. Berlin: Springer-Verlag, 1968, pp. 239-271.

[14] C. A. Felippa, "A historical outline of matrix structural analysis: a play in three acts," *Computers & Structures*, Vol. 79, No. 14, 2001, pp. 1313-1324.

[15] V. Dias da Silva, *Mechanics and Strength of Materials.* Berlin: Springer-Verlag, 2006.